

Structural stability for the Brinkman equations of flow in double diffusive convection

Changhao Lin^a, L.E. Payne^{b,*}

^a *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, PR China*

^b *Department of Mathematics, Cornell University, Ithaca, NY 14853, USA*

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Abstract

This paper continues the investigation of structural stability for the Brinkman equations modeling the double diffusive convection for flow in a porous medium. It supplements earlier results of Straughan and Hutter [B. Straughan, K. Hutter, A priori bounds and structural stability for double diffusive convection incorporating the Soret effect, *Proc. R. Soc. Lond. Ser. A* 455 (1999) 767–777].

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1. Introduction

In a recent paper Straughan and Hutter [12] examined the Soret effect on a double diffusive convective motion of a Brinkman fluid. In particular they derived a priori inequalities which implied the continuous dependence of the solution of a specific initial-boundary value problem on the Soret coefficient. The governing equations may be written as

$$\begin{cases} -v\Delta u_i + u_i = -p_{,i} + g_i T + h_i C \\ T_{,t} + u_i T_{,i} = \Delta T \\ C_{,t} + u_i C_{,i} = \Delta C + \sigma \Delta T \\ u_{i,i} = 0 \end{cases} \quad \text{in } \Omega \in \{t > 0\}, \quad (1.1)$$

* Corresponding author.

E-mail address: lep8@cornell.edu (L.E. Payne).

where Δ is the Laplace operator and u_i , T , C and p represent fluid velocity, temperature, salt concentration and pressure, respectively. The quantities $g_i(x)$ and $h_i(x)$ are gravity vector terms and the constant σ is the so-called Soret coefficient. In (1.1) and in the equations throughout a comma denotes differentiation and we employ the convention of summing over repeated indices from 1 to 3.

Let Ω be a bounded domain in \mathbb{R}^3 with Lipschitz boundary $\partial\Omega$. Then associated with (1.1) we impose the boundary data on $\partial\Omega$

$$u_i = 0, \quad T = f_1, \quad C = f_2 \quad (1.2)$$

with prescribed functions f_i . We also impose initial data

$$T(x, 0) = T_0(x), \quad C(x, 0) = C_0(x) \quad \text{in } \Omega. \quad (1.3)$$

In [12], the authors established continuous dependence of the solution on the coefficient σ . Here we examine other structural stability questions, i.e. the continuous dependence of the solution on ν , g_i and h_i . Many authors have recently dealt with such structural stability problems (see, e.g., [2,4,7,8,11,12] and papers cited therein).

2. Continuous dependence on the coefficients g_i and h_i

This section is devoted to establishing continuous dependence of the solution on g_i and h_i . Let (u_i, p, T, C) and (u_i^*, p^*, T^*, C^*) be two solutions of (1.1) with the same data (1.2), (1.3), but with different coefficients (g_i, h_i) and (g_i^*, h_i^*) , respectively. Now set

$$\begin{cases} w_i = u_i - u_i^*, & \pi = p - p^*, & S = T - T^*, & \Sigma = C - C^*, \\ \gamma_i = g_i - g_i^*, & \mu_i = h_i - h_i^*. \end{cases} \quad (2.1)$$

The difference of the two solutions (w_i, π, S, Σ) then satisfies

$$\begin{cases} -\nu \Delta w_i + w_i = -\pi_{,i} + \gamma_i T + g_i^* S + \mu_i C + h_i^* \Sigma \\ S_{,t} + w_i T_{,i} + u_i^* S_{,i} = \Delta S \\ C_{,t} + w_i C_{,i} + u_i^* \Sigma_{,i} = \Delta \Sigma + \sigma \Delta S \\ w_{i,i} = 0 \end{cases} \quad \text{in } \Omega \times \{t > 0\}, \quad (2.2)$$

with the boundary and initial conditions

$$\begin{cases} w_i = S = \Sigma = 0 & \text{on } \partial\Omega \times \{t > 0\}, \\ S(x, 0) = \Sigma(x, 0) = 0 & \text{in } \Omega. \end{cases} \quad (2.3)$$

Multiplying (2.2) by w_i and integrating we have

$$\int_{\Omega} w_i (-\nu \Delta w_i + w_i + \pi_{,i} - \gamma_i T - g_i^* S - \mu_i C - h_i^* \Sigma) dx = 0 \quad (2.4)$$

which leads to

$$\nu \int_{\Omega} w_{i,j} w_{i,j} dx + \int_{\Omega} w_i w_i dx = \int_{\Omega} w_i [\gamma_i T + g_i^* S + \mu_i C + h_i^* \Sigma] dx. \quad (2.5)$$

By using the arithmetic–geometric mean inequality in (2.5) we obtain

$$\begin{aligned}
& v \int_{\Omega} w_{i,j} w_{i,j} dx + \left(1 - \frac{\beta_1}{2} - \frac{\beta_2}{2} - \frac{\beta_3}{2} - \frac{\beta_4}{2}\right) \int_{\Omega} w_i w_i dx \\
& \leq \frac{\gamma^2}{2\beta_1} \int_{\Omega} T^2 dx + \frac{(g^*)^2}{2\beta_2} \int_{\Omega} S^2 dx + \frac{\mu^2}{2\beta_3} \int_{\Omega} C^2 dx + \frac{(h^*)^2}{2\beta_4} \int_{\Omega} \Sigma^2 dx,
\end{aligned} \quad (2.6)$$

where

$$\gamma^2 = \max_{\Omega} \gamma_i \gamma_i, \quad \mu^2 = \max_{\Omega} \mu_i \mu_i, \quad (g^*)^2 = \max_{\Omega} g_i^* g_i^*, \quad (h^*)^2 = \max_{\Omega} h_i^* h_i^*. \quad (2.7)$$

Now

$$\int_{\Omega} w_{i,j} w_{i,j} dx \geq \lambda \int_{\Omega} w_i w_i dx, \quad (2.8)$$

where λ is the first eigenvalue of the problem

$$\begin{cases} \Delta \varphi + \lambda \varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

Numerous lower bounds for λ are known (see, e.g., Bandle [1]). Thus we have

$$\begin{aligned}
& \left(1 + \lambda v - \frac{1}{2} \sum_{j=1}^4 \beta_j\right) \int_{\Omega} w_i w_i dx \\
& \leq \frac{\gamma^2}{2\beta_1} \int_{\Omega} T^2 dx + \frac{(g^*)^2}{2\beta_2} \int_{\Omega} S^2 dx + \frac{\mu^2}{2\beta_3} \int_{\Omega} C^2 dx + \frac{(h^*)^2}{2\beta_4} \int_{\Omega} \Sigma^2 dx.
\end{aligned} \quad (2.10)$$

Choosing all $\beta_i = \frac{1}{2}$ in (2.6) we thus obtain

$$\begin{aligned}
\lambda v \int_{\Omega} w_i w_i dx & \leq v \int_{\Omega} w_{i,j} w_{i,j} dx \\
& \leq \gamma^2 \int_{\Omega} T^2 dx + (g^*)^2 \int_{\Omega} S^2 dx + \mu^2 \int_{\Omega} C^2 dx \\
& \quad + (h^*)^2 \int_{\Omega} \Sigma^2 dx.
\end{aligned} \quad (2.11)$$

We note that from Eq. (2.2)

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} S^2 dx & = -2 \int_{\Omega} S_{,i} S_{,i} dx - 2 \int_{\Omega} w_i S T_{,i} dx \\
& = -2 \int_{\Omega} S_{,i} S_{,i} dx + 2\Lambda \left(\int_{\Omega} w_{i,j} w_{i,j} dx \right)^{1/2} \left(\int_{\Omega} S_{,i} S_{,i} dx \right)^{1/2} \\
& \quad \times \left(\int_{\Omega} T_{,i} T_{,i} dx \right)^{1/2}
\end{aligned}$$

$$\leq -2(1 - \alpha_1) \int_{\Omega} S_{,i} S_{,i} \, dx + \frac{\Lambda^2}{2\alpha_1} \int_{\Omega} w_{i,j} w_{i,j} \, dx \int_{\Omega} T_{,i} T_{,i} \, dx \quad (2.12)$$

for $\alpha_1 > 0$. In (2.12) we have used the Sobolev inequality which holds for $\varphi \in C_0^1(\varphi)$

$$\int_{\Omega} \varphi^4 \, dx \leq c_1 \left(\int_{\Omega} \varphi^2 \, dx \right)^{1/2} \left(\int_{\Omega} \varphi_{,i} \varphi_{,i} \, dx \right)^{3/2} \leq \Lambda \left(\int_{\Omega} \varphi_{,i} \varphi_{,i} \, dx \right)^2 \quad (2.13)$$

(see, e.g., Payne [5], Serrin [10]).

From Eq. (2.2), as in (2.12), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \Sigma^2 \, dx + (2 - \alpha_2 - \alpha_3) \int_{\Omega} \Sigma_{,i} \Sigma_{,i} \, dx \\ & \leq \frac{\sigma^2}{\alpha_2} \int_{\Omega} S_{,i} S_{,i} \, dx + \frac{\Lambda^2}{\alpha_3} \int_{\Omega} w_{i,j} w_{i,j} \, dx \int_{\Omega} C_{,i} C_{,i} \, dx \end{aligned} \quad (2.14)$$

for $\alpha_2, \alpha_3 > 0$. Combining (2.12) and (2.14) leads to

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (\Sigma^2 + \Gamma S^2) \, dx + (2 - \alpha_2 - \alpha_3) \int_{\Omega} \Sigma_{,i} \Sigma_{,i} \, dx \\ & + 2 \left[(1 - \alpha_1) - \frac{\sigma^2}{2\alpha_2 \Gamma} \right] \int_{\Omega} \Gamma S_{,i} S_{,i} \, dx \\ & \leq \frac{\Lambda^2 \Gamma}{2\alpha_1} \int_{\Omega} w_{i,j} w_{i,j} \, dx \cdot \int_{\Omega} T_{,i} T_{,i} \, dx + \frac{\Lambda^2}{\alpha_3} \int_{\Omega} w_{i,j} w_{i,j} \, dx \int_{\Omega} C_{,i} C_{,i} \, dx \\ & = \frac{\Lambda^2}{2} \int_{\Omega} w_{i,j} w_{i,j} \, dx \cdot \left[\frac{\Gamma}{\alpha_1} \int_{\Omega} T_{,i} T_{,i} \, dx + \frac{2}{\alpha_2} \int_{\Omega} C_{,i} C_{,i} \, dx \right]. \end{aligned} \quad (2.15)$$

Suppose that in (2.15) we choose

$$\alpha_2 = \alpha_3 = 1, \quad \alpha_1 = \frac{1}{2}, \quad \Gamma = \sigma^2. \quad (2.16)$$

Then, using (2.11), we find that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (\Sigma^2 + \sigma^2 S^2) \, dx \\ & \leq \frac{\Lambda^2}{\nu} \left[\gamma^2 \int_{\Omega} T^2 \, dx + (g^*)^2 \int_{\Omega} S^2 \, dx + \mu^2 \int_{\Omega} C^2 \, dx + (h^*)^2 \int_{\Omega} \Sigma^2 \, dx \right] \\ & \quad \times \left[\sigma^2 \int_{\Omega} T_{,i} T_{,i} \, dx + \int_{\Omega} C_{,i} C_{,i} \, dx \right] \\ & \leq M_1 \int_{\Omega} (\Sigma^2 + \sigma^2 S^2) \, dx \cdot \left[\sigma^2 \int_{\Omega} T_{,i} T_{,i} \, dx + \int_{\Omega} C_{,i} C_{,i} \, dx \right] \end{aligned}$$

$$+ \frac{\Lambda^2}{\nu} \left[\gamma^2 \int_{\Omega} T^2 dx + \mu^2 \int_{\Omega} C^2 dx \right] \cdot \left[\sigma^2 \int_{\Omega} T_{,i} T_{,i} dx + \int_{\Omega} C_{,i} C_{,i} dx \right], \quad (2.17)$$

where

$$M_1 = \frac{\Lambda^2}{\nu} \cdot \max \left[\frac{(g^*)^2}{\sigma^2}, (h^*)^2 \right]. \quad (2.18)$$

We now define

$$\omega(t) = \int_0^t \int_{\Omega} (\sigma^2 T_{,i} T_{,i} + C_{,i} C_{,i}) dx d\eta. \quad (2.19)$$

Then (2.17) may be rewritten as

$$\begin{aligned} & \frac{d}{dt} \left[\int_{\Omega} (\Sigma^2 + \sigma^2 S^2) dx \cdot e^{-M_1 \omega(t)} \right] \\ & \leq \frac{\Lambda^2}{\nu} \left[\gamma^2 \int_{\Omega} T^2 dx + \mu^2 \int_{\Omega} C^2 dx \right] \omega'(t) e^{-M_1 \omega(t)}. \end{aligned} \quad (2.20)$$

An integration of (2.20) yields

$$\begin{aligned} \int_{\Omega} (\Sigma^2 + \sigma^2 S^2) dx & \leq \frac{\Lambda^2}{\nu} e^{M_1 \omega(t)} \int_0^t \left(\gamma^2 \int_{\Omega} T^2 dx + \mu^2 \int_{\Omega} C^2 dx \right) \omega'(\eta) e^{-M_1 \omega(\eta)} d\eta \\ & \leq \frac{\Lambda^2}{\nu M_1} \max_{0 < \eta < t} \left[\gamma^2 \int_{\Omega} T^2 dx + \mu^2 \int_{\Omega} C^2 dx \right] (e^{M_1 \omega(t)} - 1). \end{aligned} \quad (2.21)$$

Now, in order to obtain the desired continuous dependence inequality we must derive a priori bounds for $\int_{\Omega} T^2 dx$, $\int_{\Omega} C^2 dx$, and $\omega(t)$. To this end we introduce the functions H , φ and ψ which are solutions of the following initial-boundary value problems, respectively.

$$\begin{cases} H_{,t} + u_i H_{,i} = \Delta H & \text{in } \Omega \times \{t > 0\}, \\ H = f_2 & \text{on } \partial\Omega \times \{t > 0\}, \\ H(x, 0) = C_0 & \text{in } \Omega, \end{cases} \quad (2.22)$$

$$\begin{cases} \varphi_{,t} - \Delta \varphi = 0 & \text{in } \Omega \times \{t > 0\}, \\ \varphi = f_1 & \text{on } \partial\Omega \times \{t > 0\}, \\ \varphi(x, 0) = T_0 & \text{in } \Omega \end{cases} \quad (2.23)$$

and

$$\begin{cases} \psi_{,t} - \Delta \psi = 0 & \text{in } \Omega \times \{t > 0\}, \\ \psi = f_2 & \text{on } \partial\Omega \times \{t > 0\}, \\ \psi(x, 0) = C_0 & \text{in } \Omega. \end{cases} \quad (2.24)$$

First we note that by multiplying Eq. (1.1) by T we obtain

$$(T^2)_{,t} + u_i (T^2)_{,i} = \Delta T^2 - 2T_{,i} T_{,i} \quad (2.25)$$

or

$$(T^2)_{,t} + u_i (T^2)_{,i} \leq \Delta T^2. \quad (2.26)$$

Consequently, by virtue of the maximum principle, we have

$$\max_{0 < \eta < t} \int_{\Omega} T^2 dx \leq \theta^2(t) |\Omega|, \quad (2.27)$$

where

$$\theta^2(t) = \max \left\{ \max_{\Omega} T_0^2, \max_{\partial\Omega \times (0,t)} f_1^2 \right\} \quad (2.28)$$

and $|\Omega|$ denotes the volume of Ω . Similarly we show that

$$\max_{0 < \eta < t} \int_{\Omega} C^2 dx \leq 2\chi^2(t) |\Omega| + \sigma^2 \int_0^t \int_{\Omega} T_{,i} T_{,i} dx d\eta. \quad (2.29)$$

To establish (2.29) we note that by the triangle inequality

$$\int_{\Omega} C^2 dx \leq 2 \int_{\Omega} (C - H)^2 dx + 2 \int_{\Omega} H^2 dx, \quad (2.30)$$

where H is solution of problem (2.22). As in (2.27) we have

$$\max_{0 < \eta < t} \int_{\Omega} H^2 dx \leq \max_{0 < \eta < t} \chi^2(t) |\Omega| \quad (2.31)$$

with

$$\chi^2(t) = \max \left\{ \max_{\Omega} C_0^2, \max_{\partial\Omega \times (0,t)} f_2^2 \right\}. \quad (2.32)$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} (C - H)^2 dx &= -2 \int_{\Omega} (C - H)_{,i} (C - H)_{,i} dx - 2\sigma \int_{\Omega} (C - H)_{,i} T_{,i} dx \\ &\leq \frac{\sigma^2}{2} \int_{\Omega} T_{,i} T_{,i} dx. \end{aligned} \quad (2.33)$$

An integration of (2.33) leads to

$$\int_{\Omega} (C - H)^2 dx \leq \frac{\sigma^2}{2} \int_0^t \int_{\Omega} T_{,i} T_{,i} dx d\eta. \quad (2.34)$$

Using the notation

$$\|V\|^2 = \int_0^t \int_{\Omega} V_{,i} V_{,i} dx d\eta \quad (2.35)$$

we make use of the facts that

$$\|T\|^2 \leq 2[\|T - \varphi\|^2 + \|\varphi\|^2] \quad (2.36)$$

and

$$\|C\|^2 \leq 3[\|C - H\|^2 + \|H - \psi\|^2 + \|\psi\|^2]. \quad (2.37)$$

We first derive bounds for $\|\varphi\|^2$ and $\|\psi\|^2$ in terms of data. Following that we bound $\|T - \varphi\|^2$ and $\|H - \psi\|^2$ in terms of data, and finally we obtain the bound for $\|C - H\|^2$. This will complete the bounds for $\|T\|^2$ and $\|C\|^2$. Since the arguments for bounding $\|\psi\|$ are identical to those for bounding $\|\varphi\|$ we will show only the procedure for bounding $\|\varphi\|$.

Now

$$\|\varphi\|^2 \leq 2\|\varphi - h_1\|^2 + 2\|h_1\|^2, \quad (2.38)$$

where for each t , h_1 satisfies

$$\Delta h_1 = 0 \quad \text{in } \Omega, \quad h_1 = f_1 \quad \text{on } \partial\Omega. \quad (2.39)$$

Clearly

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (\varphi - h_1)^2 dx + \|\varphi - h_1\|^2 \\ &= \frac{1}{2} \int_{\Omega} [T_0 - h_1(x, 0)]^2 dx - \int_0^t \int_{\Omega} (\varphi - h_1) h_{1,\eta} dx d\eta. \end{aligned} \quad (2.40)$$

Since $T_0 - h_1(x, 0)$ vanishes on $\partial\Omega$ we have

$$\begin{aligned} \int_{\Omega} (T_0 - h_1)^2 dx &\leq \frac{1}{\lambda} \int_{\Omega} (T_0 - h_1)_{,i} (T_0 - h_1)_{,i} dx \\ &= -\frac{1}{\lambda} \int_{\Omega} (T_0 - h_1) \Delta T_0 dx \\ &= \frac{1}{\lambda} \int_{\Omega} (T_0 - h_1)_{,i} T_{0,i} dx \end{aligned} \quad (2.41)$$

which implies that

$$\int_{\Omega} (T_0 - h_1)^2 dx \leq \frac{1}{\lambda} \int_{\Omega} T_{0,i} T_{0,i} dx. \quad (2.42)$$

On the other hand, making use of the arithmetic–geometric mean inequality on the last term of (2.40) we conclude that

$$\begin{aligned} & \int_{\Omega} (\varphi - h_1)^2 dx + \|\varphi - h_1\|^2 \\ &\leq \frac{1}{\lambda} \int_{\Omega} T_{0,i} T_{0,i} dx + \frac{m}{\lambda} \int_0^t \oint_{\partial\Omega} [f_{1,\eta}]^2 dx d\eta := P_1(t), \end{aligned} \quad (2.43)$$

where m is the so-called Dirichlet eigenvalue, i.e. the first eigenvalue of

$$\begin{aligned} \Delta^2 V &= 0 \quad \text{in } \Omega, \\ V &= 0; \quad \Delta V - m \frac{\partial V}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (2.44)$$

Lower bounds for m may be found for instance in Kuttler and Sigillito [3] and in Payne [6].

Clearly then if h_2 satisfies

$$\Delta h_2 = 0 \quad \text{in } \Omega, \quad h_2 = f_2 \quad \text{on } \partial\Omega, \quad (2.45)$$

we have

$$\begin{aligned} &\int_{\Omega} (\psi - h_2)^2 dx + \|\psi - h_2\|^2 \\ &\leq \frac{1}{\lambda} \int_{\Omega} C_{0,i} C_{0,i} dx + \frac{m}{\lambda} \int_0^t \int_{\Omega} [f_{2,\eta}]^2 dx d\eta := P_2(t). \end{aligned} \quad (2.46)$$

Finally we make use of a well-known inequality to write

$$\|h_1\|^2 \leq k_1 \int_0^t \oint_{\partial\Omega} f_1^2 ds d\eta + k_2 \int_0^t \oint_{\partial\Omega} |\text{grad}_s f_1|^2 ds d\eta = Q_1(t), \quad (2.47)$$

$$\|h_2\|^2 \leq k_1 \int_0^t \oint_{\partial\Omega} f_2^2 ds d\eta + k_2 \int_0^t \oint_{\partial\Omega} |\text{grad}_s f_2|^2 ds d\eta = Q_2(t) \quad (2.48)$$

for computable k_1 and k_2 (see [9]).

We next note that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (T - \varphi)^2 dx + \|T - \varphi\|^2 &= - \int_0^t \int_{\Omega} (T - \varphi)_{,i} u_i \varphi dx d\eta \\ &\leq \frac{1}{2} \|T - \varphi\|^2 + \frac{1}{2} \theta^2 \int_0^t \int_{\Omega} u_i u_i dx d\eta, \end{aligned} \quad (2.49)$$

and make use of the fact that

$$\begin{aligned} \int_{\Omega} u_i u_i dx &\leq 2g^2 \int_{\Omega} T^2 dx + 2h^2 \int_{\Omega} C^2 dx \\ &\leq 2[g^2\theta^2 + h^2\chi^2]|\Omega| + \sigma^2 \|T\|^2. \end{aligned} \quad (2.50)$$

Inserting (2.50) back into (2.49) we have

$$\begin{aligned} &\int_{\Omega} (T - \varphi)^2 dx + \|T - \varphi\|^2 \\ &\leq Q^2 \left\{ 2t[g^2\theta^2 + h^2\chi^2]|\Omega| + \sigma^2 \int_0^t \|T\|^2 d\eta \right\}. \end{aligned} \quad (2.51)$$

It now follows from (2.36), using (2.51), (2.43) and (2.47), that

$$\|T\|^2 \leq M_2 t + M_3 \int_0^t \|T\|^2 d\eta + 2P_1(t) + 2Q_1(t), \quad (2.52)$$

where

$$M_2 = 2\theta^2(g\theta^2 + 2h)^2|\Omega|; \quad M_3 = \theta^2\sigma^2. \quad (2.53)$$

But (2.52) integrates to give

$$\begin{aligned} & \int_0^t \|T\|^2 d\eta \\ & \leq \left[M_2 \int_0^t \eta e^{-M_3\eta} d\eta + 2 \int_0^t [P_1(\eta) + Q_1(\eta)] e^{-M_3\eta} d\eta \right] e^{M_3 t}. \end{aligned} \quad (2.54)$$

This can be simplified since

$$\begin{aligned} & \int_0^t P_1(\eta) d\eta \\ & = \left\{ \frac{1}{M_3 \lambda} \int_{\Omega} T_{0,i} T_{0,i} dx + \frac{1}{\lambda m M_3} \int_0^t \oint_{\partial\Omega} e^{-M_3\eta} (f_{1,\eta})^2 dS d\eta \right\} e^{M_3 t} \end{aligned} \quad (2.55)$$

and

$$\begin{aligned} \int_0^t Q_1(\eta) d\eta & = \left\{ \frac{k_1}{M_3} \int_0^t \oint_{\partial\Omega} e^{-M_3\eta} f_1^2 ds d\eta \right. \\ & \quad \left. + \frac{k_2}{M_3} \int_0^t \int_{\partial\Omega} e^{-M_3\eta} |\text{grad}_s f_1|^2 \right\} e^{M_3 t}. \end{aligned} \quad (2.56)$$

Now inserting (2.54) into (2.52) we conclude that

$$\|T\|^2 \leq 2M_2 t + 4P_1(t) + 4Q_1(t) + 2M_3 \left\{ \frac{M_2}{M_3^2} + 2 \int_0^t [P_1(\eta) + Q_1(\eta)] d\eta \right\} e^{M_3 t}. \quad (2.57)$$

The bound for $\|C\|^2$ follows directly when we note that

$$\|C\|^2 \leq 2\|C - H\|^2 + 2\|H\|^2 \quad (2.58)$$

and the fact that

$$\|C - H\|^2 \leq \sigma^2 \|T\|^2, \quad (2.59)$$

since the arguments used for determining the bound for $\|H\|^2$ are identical to those used in deriving the bound for $\|T\|^2$. Thus the insertion of the bounds for $\|T\|^2$ and $\|C\|^2$ into (2.21)

results in the coefficient of $(e^{M_1\omega(t)} - 1)$ in (2.21) being bounded in terms of data. Then making use of (2.27), (2.29) and (2.57) we obtain a bound for $\omega(t)$. It follows that the bound for $\int_{\Omega} (\Sigma^2 + \sigma^2 S^2) dx$ takes the form

$$\int_{\Omega} (\Sigma^2 + \sigma S^2) dx \leq Q_3(t)\gamma^2 + Q_4(t)\mu^2 \quad (2.60)$$

for computable Q_3 and Q_4 . For finite t then (2.60) establishes continuous dependence on the coefficients g_i and h_i .

3. Continuous dependence on the viscosity coefficient ν

In this section we demonstrate briefly how to establish a continuous dependence result for the effective viscosity ν in (1.1)–(1.3). Let (u_i, p, T, C) and (u_i^*, p^*, T^*, C) be two solutions of problem (1.1)–(1.3) for different viscosity coefficients ν_1 and ν_2 , respectively. Then, as previously, (w_i, π, S, Σ) will solve the problem

$$\begin{cases} -(\nu_1 - \nu_2)\Delta u_i - \nu_2\Delta w_i + w_i = -\pi_{,i} + g_i S + h_i \Sigma, \\ S_{,t} + w_i T_{,i} + u_i^* S_{,i} = \Delta S, \\ \Sigma_{,t} + w_i C_{,i} + u_i^* \Sigma_{,i} = \Delta \Sigma + \sigma \Delta S \end{cases} \quad (3.1)$$

subject to conditions

$$w_i = S = \Sigma = 0 \quad \text{on } \partial\Omega \times \{t > 0\}, \quad (3.2)$$

$$S(x, 0) = \Sigma(x, 0) = 0 \quad \text{in } \Omega. \quad (3.3)$$

Multiplying (3.1) by w_i and integrating over Ω , results in

$$\begin{aligned} & \nu_2 \int_{\Omega} w_{i,j} w_{i,j} dx + \int_{\Omega} w_i w_i dx \\ & \leq 2g^2 \int_{\Omega} S^2 dx + 2h^2 \int_{\Omega} \Sigma^2 dx + \frac{(\nu_1 - \nu_2)^2}{\nu_2} \int_{\Omega} u_{i,j} u_{i,j} dx. \end{aligned} \quad (3.4)$$

But, from Eq. (1.1)₁ we have

$$\nu_1 \int_{\Omega} u_{i,j} u_{i,j} dx \leq \left[g^2 \int_{\Omega} T^2 dx + h^2 \int_{\Omega} C^2 dx \right]. \quad (3.5)$$

Combining (3.4) and (3.5) we obtain

$$\begin{aligned} & \nu_2 \int_{\Omega} w_{i,j} w_{i,j} dx + \int_{\Omega} w_i w_i dx \\ & \leq 2g^2 \int_{\Omega} S^2 dx + 2h^2 \int_{\Omega} \Sigma^2 dx + \frac{(\nu_1 - \nu_2)^2}{\nu_1 \nu_2} \left[g^2 \int_{\Omega} T^2 dx + h^2 \int_{\Omega} C^2 dx \right]. \end{aligned} \quad (3.6)$$

From symmetry we also obtain

$$\begin{aligned} & \nu_1 \int_{\Omega} w_{i,j} w_{i,j} dx + \int_{\Omega} w_i w_i dx \\ & \leq 2g^2 \int_{\Omega} S^2 dx + 2h^2 \int_{\Omega} \Sigma^2 dx + \frac{(\nu_1 - \nu_2)^2}{\nu_1 \nu_2} \left[g^2 \int_{\Omega} T^{*2} dx + h^2 \int_{\Omega} C^{*2} dx \right]. \end{aligned} \quad (3.7)$$

It follows then that

$$\begin{aligned} & \int_{\Omega} w_{i,j} w_{i,j} dx \\ & \leq \frac{1}{\nu_1 + \nu_2} \left[2g^2 \int_{\Omega} S^2 dx + 2h^2 \int_{\Omega} \Sigma^2 dx \right] \\ & \quad + \frac{(\nu_1 - \nu_2)^2}{\nu_1 \nu_2 (\nu_1 + \nu_2)} \left[g^2 \int_{\Omega} (T^2 + T^{*2}) dx + h^2 \int_{\Omega} (C^2 + C^{*2}) dx \right]. \end{aligned} \quad (3.8)$$

Since $\int_{\Omega} T^{*2} dx$ and $\int_{\Omega} T^2 dx$, $\int_{\Omega} C^{*2} dx$ and $\int_{\Omega} C^2 dx$ have the same a priori bound, (3.8) may be reduced to

$$\begin{aligned} & \int_{\Omega} w_{i,j} w_{i,j} dx \\ & \leq \frac{1}{\nu_1 + \nu_2} \left[2g^2 \int_{\Omega} S^2 dx + 2h^2 \int_{\Omega} \Sigma^2 dx \right] \\ & \quad + \frac{2(\nu_1 - \nu_2)^2}{\nu_1 \nu_2 (\nu_1 + \nu_2)} \left[g^2 \int_{\Omega} T^2 dx + h^2 \int_{\Omega} C^2 dx \right]. \end{aligned} \quad (3.9)$$

We note that (3.9) is analogous to (2.11), so that by employing arguments similar to those used in the previous section we can establish inequalities that imply continuous dependence of solutions of (1.1)–(1.3) on the coefficient ν , i.e. an inequality of the form

$$\int_{\Omega} (\Sigma^2 + \sigma^2 S^2) dx \leq \frac{(\nu_1 - \nu_2)^2}{\nu_1 \nu_2 (\nu_1 + \nu_2)} Q_5(t), \quad (3.10)$$

where $Q_5(t)$ is a data term. A continuous dependence estimates analogous to (3.10) for integral $\int_{\Omega} w_{i,j} w_{i,j} dx$ is obtained from (3.9). We thus conclude that for nonzero ν the solutions of Brinkman fluid equations depend continuously on the effective viscosity coefficient.

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